BRST operator quantization of generally covariant gauge systems

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Abstract

The BRST generator is realized as a Hermitian nilpotent operator for a finite-dimensional gauge system featuring a quadratic super-Hamiltonian and linear supermomentum constraints. As a result, the emerging ordering for the Hamiltonian constraint is not trivial, because the potential must enter the kinetic term in order to obtain a quantization invariant under scaling. Namely, BRST quantization does not lead to the curvature term used in the literature as a means to get that invariance. The inclusion of the potential in the kinetic term, far from being unnatural, is beautifully justified in light of the Jacobi's principle.

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I. INTRODUCTION

The gravitational field is a generally covariant system with a Hamiltonian which is constrained to vanish. Actually, the "geometrodynamical" Hamiltonian is a linear combination of four constraints (in each space point); three of them are the supermomenta (linear and homogeneous functions of the field momenta), and the other one is the super-Hamiltonian (a quadratic function in the field momenta). The quantization of such a system requires searching for a factor ordering that leads to constraint operators preserving the algebra of constraints (absence of anomalies). This issue is unsolved yet [1,2]. In order to deal with a more simple system, it is common to freeze most degrees of freedom to end with a finite-dimensional system, featuring constraints that resemble those of the gravitational field (*minisuperspace* models). In this spirit, Hájíček and Kuchař [3] have studied the quantization of one such finite dimensional system in the context of Dirac quantization.

In addition to the Dirac method, the Becci-Rouet-Stora-Tyutin (BRST) formalism is a powerful tool to quantize a first class constrained system. BRST method is based on the realization of the generator of a rigid supersymmetry, the BRST symmetry, as a Hermitian nilpotent operator, the physical quantum states being picked up from the cohomology of this operator. The power of the BRST formalism consists in the automatic invariance of the quantization under combinations of constraints, because these combinations are equivalent to coordinate changes in the fermionic sector.

Our aim is to perform the BRST quantization of a system such us the one studied in Ref. [3], i.e., a generally covariant system described by n canonical pairs (q^i, p_i) subjected to $m + 1$ first class constraints, where m of them are linear and homogeneous in the momenta, and the other one is a quadratic function of the momenta, with an indefinite nondegenerate metric G^{ij}, plus a nonvanishing potential V.

The paper is organized as follows. In Sec. II we review the BRST quantization for a system of linear constraints. It is emphasized that the usual ghost contribution to the constraint operator (the anti-Hermitian term $\frac{i}{2}C_{ab}^b$) is related to the volume density induced

by the constraints on the gauge orbit. In Sec. III we add a quadratic constraint. We take advantage of the possibility of factorizing out the nonvanishing potential in the Hamiltonian constraint; this is equivalent to scale the constraint for obtaining an equivalent one with a constant potential. So, we first get the Hermitian nilpotent BRST generator $\hat{\Omega}$ for a constant potential, and then the original potential is reentered by means of a unitary transformation. The volume density of Sec. II plays an essential and elegant role in guessing the ordering that leads to the nilpotency. In Sec. IV we look for the constraint operators of the Dirac method. In the BRST formalism they can be got from $\hat{\Omega}$ after a suitable ordering of the ghost sector. As it happens with the linear constraints, the Hamiltonian constraint also gets a non-Hermitian ghost contribution. Section V is devoted to the conclusions. We emphasize the role played by the ghost contributions to the constraint operators in preserving the invariance of the theory under the relevant combinations of constraints (those which do not change the form of the constraints). We reduce the system by fixing the gauge freedom coming from the linear constraints, and we show that the ghost contribution to the super-Hamiltonian leads to the emerging of the Laplacian associated with the scale-invariant metric $V G_{ij}$, in a beautiful agreement with the Jacobi's principle. Namely, no curvature term is needed to get the invariance under scaling.

II. LINEAR CONSTRAINTS

For a system of m linearly independent constraints

$$G_a(q^i, p_j) = \xi_a^k(q^i)p_k, \qquad a = 1, ..., m, \tag{2.1}$$

the problem of finding a factor ordering satisfying the algebra

$$\{G_a, G_b\} = C_{ab}^c(q)G_c \tag{2.2}$$

is trivially solved by

$$\hat{G}_a = f^{\frac{1}{2}}\xi_a^i \hat{p}_i f^{-\frac{1}{2}}, \tag{2.3}$$

where f is arbitrary. In the Dirac quantization the function f can be determined by asking the constraint operators to preserve the geometrical character of the wave function [3–5]. This character is determined by the transformation law of the wave function under the changes that leave invariant the classical theory: coordinate changes and linear combinations of constraints. The wave function should change in such a way that the physical inner product remains unchanged.

On the other hand, in the BRST formalism, the original phase space is extended by including a canonically conjugate pair of ghost $(\eta^a, \mathcal{P}_a)$ for each constraint, with opposite parity. The central object is the BRST generator, a fermionic function $\Omega = \Omega(q^i, p_j, \eta^a, \mathcal{P}_b)$ that captures all the identities satisfied by the system of first class constraints in the equation

$$\{\Omega, \Omega\} = 0. \tag{2.4}$$

The existence of Ω is guaranteed at the classical level, and Ω is unique up to canonical transformations in the extended phase space. It can be built by means of a recursive method [6]. The result for the system (2.1) and (2.2) is

$$\Omega^{linear} = \eta^a G_a + \frac{1}{2}\eta^a\eta^b C_{ab}^c \mathcal{P}_c. \tag{2.5}$$

In order to quantize the extended system, the classical BRST generator must be realized as a Hermitian operator. The theory is free from BRST anomalies, if a Hermitian realization of Ω can be found such that the classical property (2.4) becomes

$$[\hat{\Omega}, \hat{\Omega}] = 2\hat{\Omega}^2 = 0, \tag{2.6}$$

i.e., $\hat{\Omega}$ must be nilpotent. The BRST physical quantum states belong to the set of equivalence classes of BRST-closed states ($\hat{\Omega}\psi = 0$) moduli BRST-exact ones ($\psi = \hat{\Omega}\chi$) (quantum BRST cohomology).

Let us adopt the notation used in Ref. [5]:

$$\eta^{c_s} = (q^i, \eta^a), \qquad \mathcal{P}_{c_s} = (p_i, \mathcal{P}_a), \qquad s = -1, 0. \tag{2.7}$$

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Then, Ω^{linear} can be written as

$$\Omega^{linear} = \sum_{s=-1}^{0} \Omega^{c_s} \mathcal{P}_{c_s}, \tag{2.8}$$

where

$$\Omega^{c_s} \equiv (\eta^a \xi_a^i \,, \; \frac{1}{2} \eta^a \eta^b C_{ab}^c). \tag{2.9}$$

The ordering

$$\hat{\Omega}^{linear} = \sum_{s=-1}^{0} f^{\frac{1}{2}} \Omega^{c_s} \hat{\mathcal{P}}_{c_s} f^{-\frac{1}{2}} \tag{2.10}$$

is nilpotent for any $f(q)$ [it is just the classical result (2.4)] but f should be chosen in such a way that $\hat{\Omega}^{linear}$ is Hermitian. It results that f must satisfy

$$C_{ab}^b = f^{-1}(f\xi_a^i)_{,i}. \tag{2.11}$$

The obtained $\hat{\Omega}^{linear}$ could be also obtained by symmetrizing Eq. (2.8). This realization of $\hat{\Omega}^{linear}$ leads to constraint operators that coincide with the ones obtained in the geometrical Dirac method (see, for example, Ref. [5] and references therein):

$$\hat{G}_a = f^{\frac{1}{2}} \xi_a^i \hat{p}_i f^{-\frac{1}{2}} = \xi_a^i \hat{p}_i - \frac{i}{2} \xi_{a,i}^i + \frac{i}{2} C_{ab}^b. \tag{2.12}$$

Although Eq. (2.11) is all one needs to establish $\hat{\Omega}^{linear}$, it does not univocally define f. In fact, the right hand-side does not change if f is multiplied by a gauge-invariant function. The following proposition will make clear the geometrical meaning of f in Eq. (2.11).

Proposition. For a given set (2.1) and (2.2), let $\tilde{\alpha}$ be a volume induced by the constraints in the original configuration space M:

$$\tilde{\alpha} \equiv \tilde{E}^1 \wedge ... \wedge \tilde{E}^m \wedge \tilde{\omega}, \tag{2.13}$$

where $\{\tilde{E}^a\}$ is the dual basis of $\{\vec{\xi}_a\}$ in $T_{\parallel}M$, the longitudinal tangent space; and $\tilde{\omega} = \omega(y) \, dy^1 \wedge ... \wedge dy^{n-m}$ is a closed $n-m$ form, the y^r's being $n-m$ functions which are left

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invariant by the gauge transformations generated by the linear constraints,[1] i.e., $dy^r(\vec{\xi}_a) = 0 \ \forall r, a$. $\tilde{\alpha}$ is the volume induced by the constraints in the gauge orbit, times a (nonchosen) volume in the "reduced" space. Then,

$$C^b_{ab} = div_{\tilde{\alpha}} \ \vec{\xi}_a. \tag{2.14}$$

Proof. We will take advantage of the fact that any basis can be (locally) Abelianized. So, we will prove the proposition for an Abelian basis, and then we will transform both sides of Eq. (2.14) showing that they remain equal for an arbitrary basis of $T_{||}M$.

Let $\{\vec{\xi}'_a\}$ be an Abelian basis in $T_{||}M$, then the left-hand side of Eq. (2.14) is $C'^b_{ab} = 0$. On the other hand, the $\tilde{\alpha}'$ divergence of a vector field $\vec{\xi}'_a$ is written, by definition [7], in terms of the exterior derivative of the $(n-1)$-form $\tilde{\alpha}'(\vec{\xi}'_a)$:

$$(div_{\tilde{\alpha}'}\vec{\xi}'_a)\tilde{\alpha}' \equiv d[\tilde{\alpha}'(\vec{\xi}'_a)]. \tag{2.15}$$

The right-hand side of Eq. (2.14) is also zero because $\tilde{\alpha}'(\vec{\xi}'_a)$ is closed. In fact, the forms $\tilde{E}'^a$ are (locally) exact, since an Abelian basis is a coordinate basis. Then, Eq. (2.14) is proved for Abelianized constraints.

Now, let us change the basis

$$\vec{\xi}_a = A_a{}^b(q) \ \vec{\xi}'_b, \qquad \tilde{E}^a = A^a{}_b(q) \ \tilde{E}'^b \tag{2.16}$$

($A^a{}_b$ being the matrix inverse to $A_a{}^b$). Then,

$$C^b_{ab} = E^b_i(\xi^j_b\xi^i_{a,j} - \xi^j_a\xi^i_{b,j}) = A^b{}_c(A^c_{a,j}\xi^j_b - A^c_{b,j}\xi^j_a). \tag{2.17}$$

On the other hand,

$$d[\tilde{\alpha}(\vec{\xi}_a)] = d[det A^{-1}\tilde{\alpha}'(\vec{\xi}_a)] = d[A_a{}^b det A^{-1}\tilde{\alpha}'(\vec{\xi}'_a)]$$

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[1]We do not call these functions "observables" because they will not be invariant under the action of the quadratic constraint we are going to introduce later.

$$= \sum_{b=1}^{m}(-1)^{b-1}(A_a{}^b \ detA^{-1})_{,j} \ dq^j \wedge \tilde{E}'^1 \wedge ...$$

$$... \wedge \tilde{E}'^{b-1} \wedge \tilde{E}'^{b+1} \wedge ... \wedge \tilde{E}'^m \wedge \tilde{\omega}$$

$$= (A_a{}^b \ detA^{-1})_{,j} \ A^c{}_b \ \xi^j_c \ \tilde{E}'^1 \wedge ... \wedge \tilde{E}'^m \wedge \tilde{\omega}, \qquad (2.18)$$

$(detA^{-1} \equiv detA^a{}_b)$, because only the component $dq^j(\vec{\xi}'_b) = \xi'^j_b = A^c{}_b \xi^j_c$ contributes.

Therefore

$$d[\tilde{\alpha}(\vec{\xi}_a)] = A^b{}_c(A^c_{a,j}\xi^j_b - A^c_{b,j}\xi^j_a)\tilde{\alpha}. \qquad (2.19)$$

Thus, Eqs. (2.17) and (2.19) tell us that both sides of Eq. (2.11) have the same value whatever the basis of $T_{||}M$ is. Then, the proposition results to be true for any set of linear and homogeneous first class constraints.

The result of the proposition means that f in Eq. (2.11) can be regarded as the component of $\tilde{\alpha}$ in the coordinate basis $\{dq^i\}$:

$$\tilde{\alpha} = f \ dq^1 \wedge ... \wedge dq^n. \qquad (2.20)$$

At the level of BRST, a redefinition of the constraints such as the one of Eq. (2.16) is regarded as a change of variables $\eta^a \rightarrow \eta'^a = \eta^b A_b{}^a(q)$. Since the BRST wave function behaves as a superdensity of weight $1/2$ in the space $(q, \ \eta)$ (in order to leave the inner product invariant), one concludes that the factors $f^{\frac{1}{2}}$ and $f^{-\frac{1}{2}}$ in Eq. (2.10) are exactly what is needed in order for $\hat{\Omega}^{linear}\psi$ to behave in the same way as ψ under such a change (in fact, $f \rightarrow f' = detA \ f$). This property of f should be taken into account at the moment of quantizing a system with a quadratic constraint, because it could facilitate the searching for the operator $\hat{\Omega}$.

III. QUADRATIC CONSTRAINT

As it was stated in the introduction, we are going to consider a quadratic constraint with a nonvanishing potential. This property enables us to factorize out the potential, and

replace the quadratic constraint by an equivalent one with a constant potential. So, let us begin by considering a Hamiltonian constraint $h(q^i, p_j)$:

$$h(q^k, p_j) = \frac{1}{2}g^{ij}(q^k)p_i p_j + \lambda, \qquad \lambda = const, \tag{3.1}$$

g^{ij} being an indefinite nondegenerate metric. A more general nonvanishing potential $V = \lambda\vartheta(q)$ will enter later.

In order that the set of constraints remains first class, we demand [together with the relations (2.2)],

$$\{h, G_a\} = c_{oa}^b(q, p)G_b, \tag{3.2}$$

where

$$c_{oa}^b(q, p) = c_{oa}^{bj}(q)p_j. \tag{3.3}$$

Since one has added a constraint, the already extended phase space must be further extended by adding the pair $(\eta^o, \mathcal{P}_o)$ associated with h. One finds that the new BRST generator is

$$\Omega = \eta^o h + \eta^a G_a + \eta^o \eta^a c_{oa}^b \mathcal{P}_b + \frac{1}{2}\eta^a \eta^b C_{ab}^c \mathcal{P}_c \equiv \Omega^{quad} + \Omega^{linear}, \tag{3.4}$$

where Ω^{linear} is the one of Eq. (2.8), and Ω^{quad} is

$$\Omega^{quad} = \frac{1}{2}\sum_{r,s=-1}^{0}\mathcal{P}_{a_r}\Omega^{a_r b_s}\mathcal{P}_{b_s} + \eta^o\lambda, \tag{3.5}$$

with

$$\Omega^{a_r b_s} \equiv \begin{pmatrix} \eta^o g^{ij} & \eta^o \eta^a c_{oa}^{bi} \\ & \\ \eta^o \eta^b c_{ob}^{aj} & 0 \end{pmatrix}. \tag{3.6}$$

One quantizes the system by turning the BRST generator in a Hermitian and nilpotent operator $\hat{\Omega}$:

$$[\hat{\Omega}, \hat{\Omega}] = [\hat{\Omega}^{quad}, \hat{\Omega}^{quad}] + 2[\hat{\Omega}^{quad}, \hat{\Omega}^{linear}] + [\hat{\Omega}^{linear}, \hat{\Omega}^{linear}] = 0. \tag{3.7}$$

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The term $[\hat{\Omega}^{quad}, \hat{\Omega}^{quad}]$ is zero trivially because $\eta^{o2} = 0$ (note that Ω does not depend on $\mathcal{P}_o$). The last term is zero because $\hat{\Omega}^{linear}$ is already nilpotent. So, we only must find an ordering for $\hat{\Omega}^{quad}$ satisfying $[\hat{\Omega}^{quad}, \hat{\Omega}^{linear}] = 0$. The structure of $\hat{\Omega}^{linear}$ strongly suggests the following Hermitian ordering for $\hat{\Omega}^{quad}$:

$$\hat{\Omega}^{quad} = \frac{1}{2} \sum_{r,s=-1}^{0} f^{-\frac{1}{2}} \hat{\mathcal{P}}_{a_r} f \Omega^{a_r b_s} \hat{\mathcal{P}}_{b_s} f^{-\frac{1}{2}} + \eta^o \lambda. \tag{3.8}$$

In fact, it is proved by direct calculation that $\hat{\Omega}$ results to be nilpotent.

IV. CONSTRAINT OPERATORS

In this section we are going to identify the Dirac constraint operators. They can be easily found by casting the Hermitian and nilpotent operator $\hat{\Omega}$, the sum of Eqs. (2.10)-(3.8), in the appropiate form. As in Sec. 14.5 of Ref. [6], we define the constraint operators of the Dirac method to be the coefficients of the ghost operators in the BRST generator written in the $\eta - \mathcal{P}$ order [i.e., all ghost momenta are put to the right of their conjugate ghost variables by using repeatedly the ghost (anti)commutation relations]:

$$\hat{\Omega} = \hat{\eta}^o (\frac{1}{2} f^{-\frac{1}{2}} \hat{p}_i g^{ij} f \hat{p}_j f^{-\frac{1}{2}} + \frac{i}{2} f^{\frac{1}{2}} c_{oa}^{aj} \hat{p}_j f^{-\frac{1}{2}} + \lambda) + \hat{\eta}^a f^{\frac{1}{2}} \xi_a^i \hat{p}_i f^{-\frac{1}{2}} +$$
$$+ \frac{1}{2} \hat{\eta}^o \hat{\eta}^a (f^{\frac{1}{2}} c_{oa}^{bj} \hat{p}_j f^{-\frac{1}{2}} + f^{-\frac{1}{2}} \hat{p}_j c_{oa}^{bj} f^{\frac{1}{2}}) \hat{\mathcal{P}}_b + \frac{1}{2} \hat{\eta}^a \hat{\eta}^b C_{ab}^c \hat{\mathcal{P}}_c. \tag{4.1}$$

This definition has the following nice properties:

(i) In the limit $\hbar \rightarrow 0$ they go over into the original classical constraints;

(ii) they satisfy the first class conditions.

In this case, the coefficient of $\hat{\eta}^o$ in the Eq. (4.1) is the quadratic constraint operator $\hat{h}$ and the coefficients of $\hat{\eta}^a$ are the supermomentum constraint operators $\hat{G}_a$.

So far we have dealt with a constant potential. The introduction of a nonvanishing potential $\lambda \vartheta(q)$ in the BRST formalism can be accomplished in a very simple way: by performing a unitary transformation

$$\hat{\Omega} \to e^{i\hat{C}} \; \hat{\Omega} \; e^{-i\hat{C}}, \tag{4.2}$$

leading to a new Hermitian and nilpotent BRST generator. So let us choose

$$\hat{C} = \frac{1}{2}[\hat{\eta}^o \; ln \; \vartheta(q) \; \hat{\mathcal{P}}_o - \hat{\mathcal{P}}_o \; ln \; \vartheta(q) \; \hat{\eta}^o], \qquad \vartheta(q) > 0. \tag{4.3}$$

Thus,

$$\hat{\Omega} = \hat{\eta}^o(\frac{1}{2}\vartheta^{\frac{1}{2}}f^{-\frac{1}{2}}\hat{p}_ig^{ij}f\hat{p}_jf^{-\frac{1}{2}}\vartheta^{\frac{1}{2}} + \frac{i}{2}\vartheta^{\frac{1}{2}}f^{\frac{1}{2}}c_{oa}^{aj}\hat{p}_jf^{-\frac{1}{2}}\vartheta^{\frac{1}{2}} + \lambda\vartheta) + \hat{\eta}^a\vartheta^{-\frac{1}{2}}f^{\frac{1}{2}}\xi_a^i\hat{p}_if^{-\frac{1}{2}}\vartheta^{\frac{1}{2}}$$
$$+\hat{\eta}^o\hat{\eta}^a\xi_a^i(ln\vartheta)_{,i}\hat{\mathcal{P}}_o + \frac{1}{2}\hat{\eta}^o\hat{\eta}^a\vartheta^{\frac{1}{2}}(f^{\frac{1}{2}}c_{oa}^{bj}\hat{p}_jf^{-\frac{1}{2}} + f^{-\frac{1}{2}}\hat{p}_jc_{oa}^{bj}f^{\frac{1}{2}})\vartheta^{\frac{1}{2}}\hat{\mathcal{P}}_b + \frac{1}{2}\hat{\eta}^a\hat{\eta}^bC_{ab}^c\hat{\mathcal{P}}_c. \tag{4.4}$$

The resulting operator $\hat{\Omega}$ corresponds to a quadratic constraint $H = \vartheta \; h$ (then, $C_{oa}^{bj} = \vartheta \; c_{oa}^{bj}$). The constraint operators can be read in Eq. (4.4):

$$\hat{H} = \frac{1}{2}\vartheta^{\frac{1}{2}}f^{-\frac{1}{2}}\hat{p}_ig^{ij}f\hat{p}_jf^{-\frac{1}{2}}\vartheta^{\frac{1}{2}} + \frac{i}{2}\vartheta^{-\frac{1}{2}}f^{\frac{1}{2}}C_{oa}^{aj}\hat{p}_jf^{-\frac{1}{2}}\vartheta^{\frac{1}{2}} + \lambda\vartheta, \tag{4.5}$$

$$\hat{G}_a = \vartheta^{-\frac{1}{2}}f^{\frac{1}{2}}\xi_a^i\hat{p}_if^{-\frac{1}{2}}\vartheta^{\frac{1}{2}}, \tag{4.6}$$

with the corresponding set of structure functions,

$$\hat{C}_{oa}^o = \xi_a^i(ln \; \vartheta)_{,i}, \tag{4.7}$$

$$\hat{C}_{oa}^b = \frac{1}{2}\left(\vartheta^{-\frac{1}{2}}f^{\frac{1}{2}}C_{oa}^{bj}\hat{p}_jf^{-\frac{1}{2}}\vartheta^{\frac{1}{2}} + \vartheta^{\frac{1}{2}}f^{-\frac{1}{2}}\hat{p}_jC_{oa}^{bj}f^{\frac{1}{2}}\vartheta^{-\frac{1}{2}}\right), \tag{4.8}$$

$$\hat{C}_{ab}^c = C_{ab}^c, \tag{4.9}$$

all of them properly ordered for satisfying of the constraint algebra,

$$[\hat{H}, \hat{G}_a] = \hat{C}_{oa}^o\hat{H} + \hat{C}_{oa}^b(q, p)\hat{G}_b, \tag{4.10}$$

$$[\hat{G}_a, \hat{G}_b] = \hat{C}_{ab}^c(q)\hat{G}_c. \tag{4.11}$$

V. CONCLUSIONS

The result (4.5) says that the operator associated with a first class constraint $H = \frac{1}{2}G^{ij}(q)p_ip_j + V(q)$, with $V(q) > 0 \quad \forall q$, is not the Laplacian for the metric G^{ij} plus V, but

$$\hat{H} = \frac{1}{2}V^{\frac{1}{2}}f^{-\frac{1}{2}}\hat{p}_iV^{-1}G^{ij}f\hat{p}_jf^{-\frac{1}{2}}V^{\frac{1}{2}} + \frac{i}{2}V^{-\frac{1}{2}}f^{\frac{1}{2}}C^{aj}_{oa}\hat{p}_jf^{-\frac{1}{2}}V^{\frac{1}{2}} + V, \tag{5.1}$$

since the metric in the kinetic term must be $g_{ij} = VG_{ij}$. For the sake of simplicity, we use a definite positive potential, but it should be noted that, in general, what is required is a nonvanishing one.

As it is well known, the BRST formalism provides ghost contributions to the constraint operators, which are needed for the satisfying of the algebra and/or for preserving the geometrical character of the wave function. The ghost contribution to the quadratic constraint is the second term in Eq. (5.1) that will be analyzed below. The linear constraints acquire two anti-Hermitian terms associated with the traces of the structure functions:

$$\hat{G}_a = V^{-\frac{1}{2}}f^{\frac{1}{2}}\xi^i_a\hat{p}_if^{-\frac{1}{2}}V^{\frac{1}{2}} = \xi^i_a\hat{p}_i - \frac{i}{2}\xi^i_{a,i} + \frac{i}{2}C^b_{ab} + \frac{i}{2}C^o_{ao}, \tag{5.2}$$

where $\frac{i}{2}C^o_{ao} = -\frac{i}{2}\xi^i_a(\ln V)_{,i}$ is the "cocycle" of Ref. [3].[2] Then, the kinetic term in the super-Hamiltonian and the supermomenta are sensible to the existence of a potential.

The Hamiltonian constraint operator (5.1) differs from the one employed in Ref. [3], where a curvature term was introduced to retain the invariance of the theory under scaling. Instead, the invariance under scaling is provided by the role played by the potential in the

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[2]Actually the restriction on the potential can be relaxed, because the results of Sec. III and IV do not change if λ, instead of being constant, is a function $\lambda(y)$ invariant on the gauge orbits associated with the linear constraints. The potential should be only restricted to factorizing as $V = \vartheta(q)\lambda(y)$, $\vartheta(q) > 0$. This factorization allows the existence of a globally well-defined "physical gauge" in Ref. [3]: $\Omega(q)$ therein could be taken to be $\ln\vartheta(q)$. However, nonpositive definite potentials make less evident the way to build the inner product in the physical Hilbert space [11].

constraint operators. In fact, the role played by the factors $f^{\pm\frac{1}{2}}$, $V^{\pm\frac{1}{2}}$ is clear whenever one pays attention to the transformations which should leave invariant the theory; these are (i) coordinate changes, (ii) combinations of the supermomenta [Eq. (2.16)], and (iii) scaling of the super-Hamiltonian ($H \rightarrow e^{\Theta} H$). The physical gauge-invariant inner product of the Dirac wave functions,

$$(\varphi_1, \varphi_2) = \int dq \; [\prod^{m+1} \delta(\chi)] \; J \; \varphi_1^*(q) \; \varphi_2(q), \tag{5.3}$$

(where J is the Faddeev-Popov determinant and χ are the $m + 1$ gauge conditions) must be invariant under any of these transformations. On account of the change of the Faddeev-Popov determinant under (ii) and (iii), the inner product will remain invariant if the Dirac wave function changes according to [5]

$$\varphi \rightarrow \varphi' = (det A)^{\frac{1}{2}} e^{-\frac{\Theta}{2}} \varphi. \tag{5.4}$$

So, the factors $f^{\pm\frac{1}{2}}$, $V^{\pm\frac{1}{2}}$ in the constraint operators are just what are needed in order that $\hat{G}_a\varphi$, $\hat{H}\varphi$, and $\hat{C}_{oa}^b\varphi$ transform as φ, so preserving the geometrical character of the Dirac wave function.

Whenever the reader prefers to regard the wave function as invariant under the relevant transformations (i)-(iii), he/she should perform the transformation

$$\varphi \rightarrow \phi = f^{-\frac{1}{2}}V^{+\frac{1}{2}}\varphi, \tag{5.5}$$

$$\hat{\mathcal{O}} \rightarrow f^{-\frac{1}{2}}V^{+\frac{1}{2}}\hat{\mathcal{O}}f^{+\frac{1}{2}}V^{-\frac{1}{2}}. \tag{5.6}$$

The corresponding physical inner product results in the integration of the invariant $\phi_1^*\phi_2$ in the invariant volume $V^{-1} \; J \; [\prod \delta(\chi)] \; \tilde{\alpha}$.

$$(\phi_1, \phi_2) = (\varphi_1, \varphi_2) = \int \tilde{\alpha} \; V^{-1} \; J \; [\prod \delta(\chi)] \; \phi_1^* \; \phi_2. \tag{5.7}$$

Since the inner product (5.3) or (5.7) is invariant under the transformation (ii), one can choose the Abelian coordinate basis $\vec{\xi}_a' = \partial/\partial Q^a$ ($G_a' = P_a$). Thus, the volume reads

$$\tilde{\alpha}' = dQ^1 \wedge ... \wedge dQ^m \wedge \omega(y)dy^1 \wedge ...dy^{n-m}. \tag{5.8}$$

Then, the linear constraint equations for the invariant Dirac wave function ϕ are

$$\frac{\partial \phi}{\partial Q^a} = 0. \tag{5.9}$$

These equations notably simplify the super-Hamiltonian constraint equation which, when written in the coordinate basis $\{dQ^a, dy^r\}$ [then $f' = \omega(y)$], reduces to

$$\left(-\frac{1}{2}V\frac{\partial}{\partial Q^a} \ V^{-1}G^{ar}\frac{\partial}{\partial y^r} - \frac{1}{2}V\omega(y)^{-1}\frac{\partial}{\partial y^r} \ \omega(y)V^{-1}G^{rs}\frac{\partial}{\partial y^s} + \frac{1}{2}C^{ar}_{oa}\frac{\partial}{\partial y^r} + V \right) \phi = 0. \tag{5.10}$$

The potential can be factorized out. Then, taking into account the Eqs. (3.2) and (3.3), it is $V^{-1}C^{br}_{oa} = c^{br}_{oa} = \partial g^{br}/\partial Q^a$ and $V^{-1}G^{rs} = g^{rs} = g^{rs}(y)$. Thus,

$$\left(-\frac{1}{2}\omega(y)^{-1}\frac{\partial}{\partial y^r} \ \omega(y)g^{rs}(y)\frac{\partial}{\partial y^s} + 1 \right) \phi = 0. \tag{5.11}$$

Therefore, the ghost contribution to the super-Hamiltonian allows for the emerging of a "Laplacian" in terms of the reduced variables $\{y^r\}$. In order to obtain the true Laplacian for the scale-invariant reduced metric $g_{rs}(y)$, one should choose $\omega(y)$ to be $|\det(g_{rs})|^{1/2}$. It is clear that the BRST formalism cannot give a value for $\omega(y)$, because $\hat{\Omega}$ is Hermitian and nilpotent whatever $\omega(y)$ is.[3]

In order to glance at the relationship between Dirac quantization and reduced space quantization, let us choose the gauge-fixing functions $\chi^a = Q^a$, $\{\chi^o, P_a\} = 0$. Then, one integrates the Q^a's in Eq. (5.7) using the volume $\tilde{\alpha}'$ to obtain

$$(\phi_1, \phi_2) = \int \tilde{\omega} \ V^{-1} \ J_o \ \delta(\chi^o) \ \phi_1^*[q^i(Q^a = 0, y^r)] \ \phi_2[q^i(Q^a = 0, y^r)]. \tag{5.12}$$

One can define a density of weight $1/2$ under changes of the y^r's: [4]

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[3]If, for instance, the system only had a quadratic constraint h, then $\hat{\Omega} = \hat{\eta}\hat{h}$ would be Hermitian and nilpotent for any Hermitian ordering of $\hat{h}$.

[4]The function $\omega^{-1}f'$ plays the role of μ in Ref. [5] and of $\mathcal{M}$ in Ref. [8] . In fact, let us use the Abelianized basis $\vec{\xi}'_a$ in $T_{\parallel}M$ which is a coordinate basis: $\vec{\xi}'_a = \partial/\partial Q^a$ $(G'_a = P_a)$, $\tilde{E}'^a = dQ^a$. Then, $\omega^{-1}f'$ is the Jacobian of the coordinate change $(Q^a, y^r) \rightarrow q^i$.

$$\varphi_R(y) = \omega(y)^{\frac{1}{2}} \, V[q^i(Q^a = 0, y^r)]^{-\frac{1}{2}} \, \phi[q^i(Q^a = 0, y^r)]$$

$$= \omega(y)^{\frac{1}{2}} \, f'[q^i(Q^a = 0, y^r)]^{-\frac{1}{2}} \, \varphi[q^i(Q^a = 0, y^r)]. \tag{5.13}$$

Then,

$$(\phi_1, \phi_2) = (\varphi_{R_1}, \varphi_{R_2}) = \int dy \, J_o \, \delta(\chi^o) \varphi_{R_1}^*(y) \varphi_{R_2}(y), \tag{5.14}$$

and φ_R is the Dirac wave function in a "reduced" space where only the quadratic constraint remains: φ_R is constrained by the Eq. (5.11) satisfied by $\phi[q^i(Q^a = 0, y^r)]$.

We close the conclusions by giving a beautiful classical argument supporting the inclusion of the potential in the kinetic term. Generally covariant systems are invariant under changes of the parameter in the functional action [6]. This means that the parameter is physically irrelevant: it is not the time. The time could be hidden among the dynamical variables and, as a result, the Hamiltonian is constrained to vanish [9]. In this case the time would be identified as a function $t(q, p)$ in phase space that monotonically increases on all the dynamical trajectories. Since the here-studied H is equivalent to a super-Hamiltonian with a constant potential, the systems embraced in this article are those resembling a relativistic particle in a curved spacetime. Then, the time is hidden in the configuration space (*intrinsic time* [9]). This means that the trajectory in the configuration space contains all the dynamical information about the system. In classical mechanics, the Jacobi's principle [10] is the variational principle for getting the paths in configuration space, for a fixed energy E, without information about the evolution of the system in the parameter of the functional action. The paths are obtained by varying the functional

$$I = \int_{q'}^{q''} \sqrt{2|E - V|G_{ij} dq^i dq^j}. \tag{5.15}$$

In our case, the energy is zero, and Eq. (5.15) looks such as the functional action of a relativistic particle of mass unity in a curved background with metric $2VG_{ij} = 2g_{ij}$. The paths are geodesics of this metric $2g_{ij}$, instead of G_{ij}. When the gauge is fixed to be $Q^a = 0$, the Jacobi's principle reduces to the variation of the functional

$$I_R = \int_{y'}^{y''} \sqrt{2VG_{rs}dy^r dy^s},$$ (5.16)

so giving the classical support to the constraint equation (5.11), where the Laplacian is the one associated with the scale-invariant metric $2VG_{rs} = 2g_{rs}(y)$ appearing in Eq. (5.16).

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